Laboratori Nazionali di Frascati

LNF-66/39

E. Ferlenghi, C. Pellegrini and B. Touschek : THE TRANSVERSE RESISTIVE-WALL INSTABILITY OF EXTREMELY RELATIVISTIC BEAMS OF ELECTRONS AND POSITRONS.

Nuovo Cimento 44, 253 (1966).

IL NUOVO CIMENTO Vol. XLIV B, N. 2 11 Agosto 1966

The Transverse Resistive-Wall Instability of Extremely Relativistic Beams of Electrons and Positrons.

E. FERLENGHI, C. PELLEGRINI and B. TOUSCHEK

Laboratori Nazionali di Frascati, C.N.E.N. - Frascati
Istituto Nazionale di Fisica Nucleare - Sezione di Roma

(ricevuto l'11 Novembre 1965)

Summary. — The transverse resistive-wall instability is discussed with special regard to the difficulties which it may cause in the performance of colliding beam experiments. Results previously obtained by Laslett, Neil and Sessler (4) as well as by Courant and Sessler (9) and by Pellegrini and Sessler (8) are confirmed. The treatment is based on a simplified geometry in which the motion of the beams is rectilinear. Application to a circular geometry is made possible by the introduction of conditions of periodicity. The results obtained apply to continuous as well as to bunched beams. The resisting-wall force between two colliding beams is also discussed.

1. – Introduction.

In the following we give a brief account of the physical principles underlying the transverse instability of high-energy beams, the discovery of which at Stanford and MURA (1-4), has raised many doubts about the feasibility of

(1) C. P. CURTIS, *et al.*: *Beam experiments with the* MURA 50 MeV FFAG *accelerator*, in *Proceedings of the International Conference on High-Energy Accelerators* (Dubna, 1963), p. 620.

(2) F. E. MILLS and G. K. O'NEIL: *Vertical instabilities in electron storage rings*, in *Proceedings of the Brookhaven Summer Study on Storage Rings, Accelerators and Experimentation at Super-High Energies* (BNL-7534, 1963), p. 368, 375.

(3) M. Q. BARTON, J. COTTINGHAM and A. TRANIS: *Rev. Sci. Instr.*, **35**, 624 (1964).

(4) L. J. LASLETT, V. K. NEIL and A. M. SESSLER: *Rev. Sci. Instr.*, **36**, 436 (1965); quoted as L.N.S.

an experimentation with colliding beams. After the meeting in Novosibirsk in March 1965 at which most participants were impressed by the urgency of coming rapidly to a full understanding of the phenomenon, intensive work was started and much correspondence exchanged particularly between the interested parties in California, Italy and at the Brookhaven National Laboratories. A summer study group organized at Stanford did much to clarify the situation, so that today it can be said that the transverse instability observed in Stanford is indeed due to the finite resistivity of the walls of the acceleration chamber and that it can be cured either by the application of a feedback or by the introduction of nonlinearities, which inhibit the build up of large amplitudes of betatron oscillations.

It is not the intention of the present note to give a review of all the work (starting from a remarkable contribution by MAXWELL [5]) which has been done on this subject. We shall rather give a description of what can now be considered the relevant part of a line of investigation, which was started by the Adone group in 1964 [6]. Most of the foundations of this work were laid independently of the work of other groups, so that in spite of the recent intense exchange of information, the present paper can be considered as corroborative evidence for at least part of the results which have been obtained elsewhere.

The phenomenon observed at Stanford was the build-up of vertical coherent betatron oscillations in the 500 MeV colliding beam arrangement. The rise time of this build-up is of the order of milliseconds. It was also found that what cured the instability of a single beam did not necessarily cure the instability of one beam in the presence of another. From its very discovery the phenomenon has been described by the Stanford group as a resistive-wall instability.

That the finite resistivity of the walls of the vacuum chamber can give rise to a build-up of betatron oscillations can be seen by the following intuitive argument: since one observes a build-up of betatron oscillations one has to look for the source of its energy, which is found in the radio-frequency system which keeps the longitudinal motion going. In order that longitudinal energy can be converted into transverse motion it is necessary that the particles « can sense » that they are moving longitudinally. If the walls of the vacuum vessel were ideal conductors at a constant distance all along the trajectory this would not be the case, for this arrangement would look exactly the same, whether the electron is at rest or not. If however the conductivity of the wall is

---

(5) J. C. MAXWELL: *A Treatise on Electricity and Magnetism*, vol. **2** (Oxford, 1873), p. 271.

(6) E. FERLENGHI and C. PELLEGRINI: *Transverse resistive wall instabilities of relativistic beams in circular accelerator and* e^+-e^- *storage rings*, unpublished report, (March 1965).

finite the electron will « feel the wind » of the wall moving past: there can be an exchange of longitudinal and transverse energy.

In the following we shall concentrate exclusively on this « frictional » phenomenon and pay scarce attention to the conservative forces between the particles.

Ideally the procedure to be followed would be to determine the fields subject to the boundary conditions on the walls of the vacuum chamber due to the given longitudinal and arbitrary vertical motion of a single particle and then in the presence of $N_1 + N_2$ particles integrate the mechanical equations of motion. This task is greatly facilitated if in the absence of fields the vertical motion of the electrons can be considered harmonic and small. The resulting mechanical equations will then be linear and—at least in the case of a single beam—with constant coefficients, so that the integration of the mechanical equations leads to the determination of the eigenvalues of the secular determinant. The eigenvalues then represent the frequencies of the transverse motion and their imaginary parts can—according to their sign—be interpreted either as a damping constant or as the rise time for the build-up of vertical betatron oscillations.

The actual treatment is based on a number of simplifications, which we list:

1) The longitudinal motion of electrons and positrons is rectilinear and fixed in the sense that the distances between the particles in each beam are constant.

2) Since in all storage rings electrons move in closed orbits a periodicity condition is imposed on the rectilinear motion.

3) The vertical motion of the electrons in the absence of fields is harmonic, so that the electron can be envisaged as an oscillator. All these oscillators have the some frequency ν_0.

4) The wall of the vacuum chamber is represented by an infinite plane at $y = 0$. The longitudinal motion is parallel to this plane.

5) The resistive-wall effect is calculated to the first power of $\sigma^{-\frac{1}{2}}$, where σ is the conductivity of the wall.

6) The distance a between the wall and the electron orbits is considered to be small compared to the reduced wavelength of the vertical betatron oscillations.

7) Coherent radiation damping is neglected and only the imaginary parts of the oscillation frequencies are calculated.

Though this list of simplifications may appear rather restrictive it will be seen that it separates the description of the malady from its cure. Particularly the conditions 1), 3) and 7) may give an overpessimistic picture of the effect.

5131

Since even this pessimistic picture admits a simple cure, we think that the simplifications 1) to 7) are quite justified.

The method which we have here outlined leads to the following results:

1) For a continuous beam the rise time is exactly equal to the rise time calculated by LASLETT, NEIL and SESSLER (4) for what in their paper is termed « circular geometry ».

2) The rise time for an extremely bunched beam is obtained. Stable and unstable modes are selected by a rule given earlier by COURANT (7).

3) The rise time for an arbitrary number of bunches is given. Also here we have complete agreement with the stability rules recently derived by COURANT and SESSLER (8).

4) The resistive-wall rise time for the interaction between an arbitrary number of infinitely short bunches of electrons and positrons is calculated. In agreement with recent work by PELLEGRINI and SESSLER (9) it is found that the instability can concern only the electrical centre of the two beams. The mechanical centre results to be always damped.

2. – Kinematical description of the electrons.

The motion of the electrons is entirely in the y-z-plane. The longitudinal motion of the electron is described by

$$z_k(t) = \xi_k + vt \tag{2.1}$$

and one has $k=1, 2, ..., N_1$. The motion of positrons is described by (2.1) with v replaced by $-v$ and $k=1, 2, ..., N_2$. In view of the generalization to closed orbits we limit ξ_k by

$$0 \leqslant \xi_k < u \,, \tag{2.2}$$

where u is the circumference of the machine. The vertical motion of the electrons is described by

$$y_k = a + 2\,\mathrm{Re}(\eta_k e^{-i\nu t})\,, \tag{2.3}$$

(7) E. D. COURANT: *Proceedings of the Particle Accelerator Conference, Washington*, 1965; *IEEE Trans.*, NS **12**, 550 (1965).

(8) E. D. COURANT and A. M. SESSLER: *Stanford Storage Ring Summer Study* (1965).

(9) C. PELLEGRINI and A. M. SESSLER: *Stanford Storage Ring Summer Study* (1965).

where η_k is a complex amplitude and it is assumed that $|\eta_k| \ll a$. ν is the vertical betatron frequency—at least as long as electrical forces are neglected. We will often use $\nu \simeq q\omega_0$, where ω_0 is the frequency of revolution of the electrons in the machine and q is a pure number.

The longitudinal distribution of the particles in the beam can be described in terms of the Fourier coefficients

$$Ng_r = \sum_k \exp\left[ir\,\xi_k/R\right], \tag{2.4}$$

where N is the total number of particles in the beam and R is the mean radius defined by $R = u/2\pi$.

A continuous beam—the case treated by L.N.S.— has

$$g_r = \delta_{r,0}, \tag{2.5}$$

where δ is the Kronecker symbol.

The vertical motion can be described in terms of a normal-mode analysis. The r-th normal mode is defined as

$$Y_r = \sum \eta_k \exp\left[-ir\,\xi_k/R\right]. \tag{2.6}$$

If only the r-th mode is excited and if its frequency is ν_r we will have $\eta_k = Y \exp\left[+ir\,\xi_k/R - i\nu r t\right]$. The r-th mode therefore corresponds to a wave of wavelength $\lambda\!\!\!^{-} = R/|r|$ (for $r \neq 0$), which propagates along the electron beam for $r > 0$ and against it for $r < 0$.

At this point it is necessary to observe that the « sharp » normal modes (2.6) are a consequence of the simplification 1) listed in the Introduction. The basic assumption for the persistence of normal modes of the form (2.6) is $\xi_i = \text{const}$. This condition is never satisfied in reality, since the fluctuations of the radiation loss (if there is no radio frequency) or the synchrotron oscillations (if there is one) will lead to a mixing of the position variables. This mixing will destroy the individuality of the higher modes and render the mode analysis useless for processes with a rise time which is smaller than the time required for the mixing of the modes. We note that the blurring of the normal modes is given by

$$\Delta s = s\,\Delta\xi/R, \tag{2.7}$$

where in the presence of a radiofrequency $\Delta\xi$ will be of the order b—the average amplitude of the synchrotron oscillations. The mixing time T_m can be defined as the time necessary for the step $\Delta s = 1$ and is therefore given by

$T_m = R/\Omega bs$, where Ω is the frequency of the synchrotron oscillations. The condition $t_r \ll T_m$ will restrict the application of the naive model described in the Introduction to long-wave phenomena.

In the case of several very short bunches (short compared to the circumference of the machine) it will be found useful to deal with the bunch as if it were a single particle. In this case eq. (2.4) can be replaced by the definition

$$BG_r = \sum_n \exp[ir\,\xi n/R] = \sum_n \exp[2\pi i r n/B]\,. \tag{2.8}$$

Here B is the total number of bunches in the beam, and it is assumed that the bunches are placed at

$$\xi_n = nu/B \qquad \text{with } n = 0, 1, \ldots, B-1. \tag{2.9}$$

The sums in (2.8) can be directly evaluated and give

$$G_r = \delta_{r,nB} \qquad \text{with } n = 0, \pm 1, \pm 2, \ldots. \tag{2.10}$$

In correspondence to eq. (2.6) we can introduce « macroscopic » normal modes by means of

$$H_r = \sum_k \eta_k \exp[-ir\,\xi_k/R] = \sum_k \exp[-2\pi irk/B]\eta_k \tag{2.11}$$

and it is clear that the functions H_r will be periodic, that is $H_{r+B} = H_r$. The macroscopic modes H_r do not mix, since a particle which escapes from one bunch cannot enter into another.

3. – The fields.

The determination of the fields will be carried out in two steps. We shall first calculate the electromagnetic field produced by a single electron and assuming that the conductivity of the wall is infinite. In the second step we shall determine the correction to this field produced by the finite conductivity of the wall. The approximation assumes that all the frequencies of interest are small compared to the inductivity (which is measured in units of s^{-1}).

The calculation of the field is simplified by carrying it out in the frame of reference in which the electron is at rest. In this frame the electron is represented as an oscillator of frequency $\nu' = \nu\gamma$ where $\gamma = (1-v^2)^{-\frac{1}{2}}$ and we have put $c = 1$. v is the velocity of the electrons. We assume that the oscillator is in the fixed position $(x', y', z') = (0, a, 0)$.

The y-component of the electrical current density of the electron is given by

$$J'_y = -iev'\,\delta(x')\,\delta(y'-a)\,\delta(z')\exp[-iv't']\eta + \text{c. c.} \tag{3.1}$$

c. c. denotes the complex conjugate and J'_y is the current associated with the vertical motion defined by (2.3).

For the following it is convenient to introduce the Fourier transforms $f(x', y'; k', \omega')$ of a quantity $F(x', y', t', z')$ by means of

$$2\pi F' = \int \mathrm{d}k' \int \mathrm{d}\omega' f'(x', y'; k', \omega') \exp[i(k'z' - \omega' t')]\,. \tag{3.2}$$

Applying this to the current defined by (3.1) one obtains

$$j'_y = -ie\eta\nu'\,\delta(\omega'-\nu')\,\delta(x')\,\delta(y'-a) + \dots\,, \tag{3.3}$$

where the $+\dots$ indicates the part arising from c. c. in (3.1). The four-current is determined by supplementing a time component ϱ in such a way that the equation of continuity $\operatorname{div} J + \dot\varrho = 0$ is satisfied. In this way one obtains

$$j' = -ie\eta\nu'\delta(\omega'-\nu')\left(0, 1, 0, \frac{\partial}{i\nu'\partial y'}\right)\delta(x')\sum_{\pm}\delta(y'\pm a)\,, \tag{3.4}$$

where the terms in the bracket represent the j'_x, j'_y, j'_z, ϱ' and the sum $\sum_{\pm}$ takes account of the mirror charge placed at $y' = -a$. The solutions of the four-dimensional Poisson equation

$$\Box A' = -4\pi J' \qquad \text{with} \qquad \Box = \Delta - \frac{\partial^2}{\partial t^2}\,, \tag{3.5}$$

where J' is the 4-current transform of j', now represent the 4-potential of the electromagnetic field in the region $y > 0$. This field satisfies the boundary condition $E_x = E_z = 0$ for an ideal conductor (the tangential components of the electric field vanish on the surface $y = 0$).

Carrying out the Fourier transformation (3.2) we can write

$$a'_y = a'^{(+)}_y + a'^{(-)}_y\,, \tag{3.6}$$

where the $a^{(\pm)}_y$ satisfy the differential equations

$$\left(\frac{1}{\varrho_\pm}\frac{\partial}{\partial\varrho_\pm}\varrho^\pm\frac{\partial}{\partial\varrho_\pm} + \lambda^2\right)a'^{(\pm)}_y = 4\pi ie\eta\nu'\delta(\omega'-\nu')\,\delta(x')\sum_{\pm}\delta(y'\pm a)\,. \tag{3.7}$$

λ^2 is the invariant wave number:

$$\lambda^2 = \omega'^2 - k'^2 \,, \tag{3.8}$$

a quantity which is zero for waves which propagate with the velocity of light either parallel or antiparallel to the direction of motion of the electrons, and $\varrho_\pm$ is defined by

$$\varrho_\pm = +\sqrt{(y' \pm a)^2 + x'^2} \,. \tag{3.9}$$

Equation (3.7) has to be solved observing the « Ausstrahlungsbedingung » which requires that a'_y should behave as $\exp[i|\lambda|\varrho]$ for timelike λ ($\omega'^2 > k'^2$) and as $\exp[-|\lambda|\varrho]$ for spacelike λ. Since (3.7) is the equation for the cylinder functions it follows that for timelike λ the solution must be proportional to the Hankel function $K_0^{(1)}(\lambda\varrho)$. The factor of proportionality is determined in the following way. $\delta(x')\,\delta(y' \pm a)$ on the right-hand side of (3.5) can be replaced by $\delta(\varrho_\pm)/2\pi\varrho_\pm$. For $\varrho \to 0$ the Hankel function $H_0^{(1)}$ behaves as $(2i/\pi)\log\lambda\varrho$. Substituting for the δ-function in (3.7) multiplying by $\varrho_\pm$ and integrating over it from 0 to ε, we then find that we must have

$$a'_q = \pi e\eta\nu'\,\delta(\omega' - \nu') \sum_\pm (\lambda\varrho_\pm) \,. \tag{3.10}$$

A time component φ has to be added to the vector potential so that the Lorentz condition can be satisfied. In this way one gets for the four-potential

$$a' = \pi e\eta\nu'\,\delta(\omega' - \nu')\left(0, 1, 0, \frac{\partial}{i\nu'\partial y'}\right) \sum_\pm H_0^{(1)}(\lambda\varrho_\pm) \,. \tag{3.11}$$

We note that $\sum H_0^{(1)}$ is an even function of y': the fourth component of a' vanishes on the surface $y' = 0$.

The six Fourier components of the field strength will be denoted by $e'_x \dots h'_z$. For infinite conductivity the tangential components of the electrical field e'_x and e'_z must vanish on the surface. This is indeed the case for we have

$$e'_x = -\partial\varphi'/\partial x' = 0 \,, \qquad e'_z = -ik'\varphi' = 0 \qquad \text{for } y = 0. \tag{3.12}$$

As has been explained in the Introduction the motion of a real conductor is observable to the electron and it is therefore convenient to treat the problem in the system in which the conductor is at rest. Remembering that x, y, ϱ and λ are invariant under a Lorentz transformation along the z-axis, we have because of $\omega' = \gamma(\omega - kv)$

$$a = \pi e\eta\nu\delta(\omega - vk - \nu)\left(0, 1, \frac{v}{i}\frac{\partial}{\nu\partial y}, \frac{\partial}{i\nu\partial y}\right) \sum H_0^{(1)} \,. \tag{3.13}$$

To formulate the boundary-value problem for a real conductor we need the tangential components of the magnetic field at the surface $y = 0$. These are

$$\text{(3.14)} \qquad \begin{cases} h_x = 2\pi e\eta\nu\delta(\omega - vk - \nu)\left(\dfrac{v}{i\nu}\dfrac{\partial^2}{\partial y^2} - ik\right)H_0^{(1)}(\lambda\varrho_0)\,, \\ h_z = 2\pi e\eta\nu\delta(\omega - vk - \nu)\,\dfrac{\partial}{\partial x}\,H_0^{(1)}(\lambda\varrho_0)\,, \end{cases}$$

where we have put $\varrho_0 = \sqrt{a^2 + x^2}$.

The boundary condition in the case of a finite conductivity can be expressed in the form

$$\text{(3.15)} \qquad \boldsymbol{e}_t = (\boldsymbol{n}\times\boldsymbol{h})(1-i)Z\,,$$

where $\boldsymbol{n}$ is a unit vector pointing from the conductor into the vacuum, e_t is the tangential component of the electric vector and Z is defined as

$$\text{(3.16)} \qquad Z = \sqrt{\omega/8\pi\sigma}\,.$$

The square root is taken to be positive for $\omega > 0$. Its analytical continuation as well that of the Hankel function will be discussed in the next Section.

In the present case we can write for (3.15)

$$\text{(3.17)} \qquad e_x = zh_z\,, \qquad e_x = -\,zh_x\,,$$

where we have used the abbreviation

$$\text{(3.18)} \qquad z = (1-i)\,Z\,.$$

Since for all frequencies in play Z is numerically very small, the boundary-value problem posed by (3.17) can be solved by expanding the solution in terms of Z. The zeroth approximation then corresponds to infinite conductivity and of course gives $e_x^0 = e_z^0 = 0$. In first approximation the right-hand side of (3.17) is replaced by h^0, *i.e.* the expressions (3.14). The boundary-value problem (3.17) has to be solved in the half space $y > 0$.

To obtain the first approximation we introduce the vector potential

$$\text{(3.19)} \qquad a^{(1)} = 2\pi e\eta\nu\,\delta(\omega - vk - \nu)(z/i\omega)(\alpha_x, \alpha_y, \alpha_z, 0)\,,$$

where the Lorentz condition imposes

$$\text{(3.20)} \qquad \frac{\partial\alpha_x}{\partial x} + \frac{\partial\alpha_y}{\partial y} + ik\alpha_z = 0\,.$$

5137

The boundary conditions (3.17) take the form

$$(3.21)\qquad \begin{cases} \alpha_x = \dfrac{\partial}{\partial x} H_0^{(1)}(\lambda\varrho_0)\,, \\[2ex] \alpha_z = -\left(\dfrac{v}{iv}\dfrac{\partial^2}{\partial y^2} - ik\right) H_0^{(1)}(\lambda\varrho_0)\,, \end{cases} \qquad \text{for } y = 0\,.$$

Putting

$$(3.22)\qquad \begin{cases} \alpha_x = \dfrac{\partial}{\partial x} H\,, \\[2ex] \alpha_y = \dfrac{\partial}{\partial y} H + \beta_y\,, \\[2ex] \alpha_z = \dfrac{\lambda^2}{ik} H + \beta_z\,, \end{cases}$$

where as an abbreviation we have put $H = H_0^{(1)}(\lambda\varrho_+)$, it is seen that the first terms on the right-hand side of this equation are regular in the semi-space $y > 0$, satisfy the « Ausstrahlungsbedingung » and the wave equation, as well as the Lorentz condition and the first of the boundary conditions (3.21). That they satisfy the wave equation follows from

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \lambda^2\right) H = 0\,.$$

The Lorentz condition imposes for $\boldsymbol{\beta}$

$$(3.23)\qquad \frac{\partial \beta_y}{\partial y} + ik\beta_z = 0\,.$$

Combining this with the second boundary condition (3.21), we get a boundary condition for β_y:

$$(3.24)\qquad \frac{\partial \beta_y}{\partial y} = \left(\frac{vk}{\nu}\frac{\partial^2}{\partial y^2} + \omega^2\right) H_0^{(1)}(\lambda\varrho_0) \qquad \text{for } y = 0\,.$$

The solution of the boundary-value problem is then given by

$$(3.25)\qquad \beta_y = \frac{vk}{\nu}\frac{\partial H}{\partial y} - \frac{\omega^2}{\lambda} M\,,$$

where M is defined as the integral

$$(3.26)\qquad M = \lambda \int_a^\infty \mathrm{d}b\, H_0^{(1)}\left(\lambda\sqrt{(y+b)^2 + x^2}\right)\,.$$

5133

It is easily seen that β_y defined by eq. (3.25) satisfies the wave equation and is free of singularities in the semi-space $y > 0$. This is obvious for the first term in (3.25). The second term is recognized as a solution, the sources of which are uniformly distributed along the line $-a > y > \infty$. The « Ausstrahlungsbedingung » makes sure that the integral (3.26) converges also in the case of spacelike λ.

Collecting the expressions (3.22) and (3.25) and inserting into (3.19) we get for the vector potential $a^{(1)}$ caused by the finite resistivity of the wall

$$(3.27) \qquad a^{(1)} = 2\pi e\eta\nu\delta(\omega - vk - \nu)\,(z/i\omega)\cdot \cdot\left(\frac{\partial H}{\partial x},\ \frac{\omega}{\nu}\frac{\partial H}{\partial y} - \frac{\omega^2}{\lambda}M,\ i\left(\frac{v}{\nu}\frac{\partial^2}{\partial y^2} + k\right)H,\ 0\right).$$

In the following Sections we shall deal exclusively with the effects which this field has on the motion of its sources.

4. – Forces on electrons and positrons.

The resistive field given by (3.27) is caused by what we shall assume to be an electron moving along the z-axis as $z(t)$. We have to determine the force exerted on another electron which moves as, say, $z_k(t) = \xi_k + vt$ or on a positron moving as $z_k(t) = \xi_k - vt$. The force will be given by

$$(4.1) \qquad F_y = \pm e(E_y \pm vH)\,.$$

We are only interested in the transverse force, though there will be of course a longitudinal force F_z. The fields E and H have to be evaluated at the position of the « passive » particle, *i.e.* at $x = 0$, $y = a$. Equation (3.27) allows us to determine e_y and h_x. Forming $f_y = \pm e(e_y \pm vh_x)$ and using the differential equation for the function H one obtains in this way

$$(4.2) \qquad f_y = 2\pi\eta m v\gamma\, \delta(\omega - kv - \nu)\, P(\omega)$$

with

$$(4.3) \qquad P(\omega) = \pm\frac{e^2}{m\gamma v}\,\nu\,\frac{z(\omega)}{\omega}\left[\left(\frac{\omega^2}{\nu}(1\mp v^2)\pm\frac{v^2}{2\nu a^2}\right)\frac{\partial H}{\partial y} \pm \frac{v\lambda^2}{2a\nu}H - \frac{\omega^2(\omega\pm vk)}{\lambda}M\right].$$

In this expression the argument of the Hankel functions is $2\lambda a$. m is the mass of the electrons and has been introduced to simplify the mechanical equations.

The analytical properties of $P(\omega)$ are defined in terms of the « Ausstrahlungsbedingung » and of causality. We observe that P has three branching points. The first is at $\omega = 0$ and is due to the factor $z(\omega)/\omega$. Since $z(\omega)f(\omega)$

represents the deformation which a signal $f(\omega)$ undergoes owing to the reflection from the not perfectly conductive wall, and since the assumed time-dependence is $\exp[-i\omega t]$ it follows that the cut has to be chosen in such a way that it does not hinder the deformation of the path of integration into the positive imaginary ω-plane. This is achieved by choosing the cut to extend from $\omega = 0$ to $\omega = -i\infty$.

The other two branching points of $P(\omega)$ are due to the singular behaviour of the Hankel functions at $2\lambda a = 0$. Using $\lambda^2 = \omega^2 - (\omega - \nu)^2/v^2$ (the value of k is imposed by the δ-function in (4.2)) we find for the position of these points

$$\omega_{1,2} = \nu(1 \pm v)^{-1} . \tag{4.4}$$

For very large energy one has $\omega_1 = \nu/2$ and $\omega_2 = \infty$. Transforming back to the frame of reference in which the electron is at rest it is realized that ω_1 corresponds to an electromagnetic wave which travels backwards and parallel to the direction of motion of the source particle, ω_2 corresponds to the forward wave. For most of the colliding-beam experiments now in construction the wavelength of the forward wave is of the order of 10 μm. In the interval $\omega_1 < \lambda < \omega_2$ λ is real and positive (definition of the Hankel functions). The cuts have to be arranged in such a way that outside this interval and for ω real, λ is positive imaginary. This is achieved by choosing the cut which starts at $\omega = \omega_1$ to extend into the negative imaginary ω-plane to, say, $\omega_1 - i\infty$. The cut which starts at ω_2 extends to the « north » to $\omega_2 + i\infty$.

We are now in a position to derive the equations of motion for the system of particles. We assume that the k-th particle is described by an oscillator, which in the absence of the resistive wall field, satisfies the equation of motion

$$\ddot{\eta}_k + \nu_0^2 \eta_k = 0 \qquad \text{or} \qquad (-\nu^2 + \nu_0^2)\eta_k = 0 . \tag{4.5}$$

For greater generality we consider a source particle (index i) which moves as $z_i(t) = \xi_i + vt$. We shall first treat the case of electron-electron interaction. Using eqs. (4.2) and (4.3) and chosing P to correspond to the upper sign in (4.3) (electron-electron interaction) we obtain the following system of mechanical equations:

$$(-\nu^2 + \nu_0^2)\eta_k = \sum_i \int d\omega\, P(\omega) \exp\left[-\frac{i\omega}{v}(\xi_i - \xi_k)\right] \exp\left[\frac{i\nu}{v}(\xi_i - \xi_k)\right] \eta_i , \tag{4.6}$$

for which one can also write

$$\ddot{\eta}_k + \nu_0^2 \eta_k = \sum_i \varrho\left(\frac{\xi_i - \xi_k}{v}\right) \eta_i \left(t - \frac{1}{v}(\xi_i - \xi_k)\right) , \tag{4.7}$$

5140

where we have put

$$\varrho(\tau) = \int \mathrm{d}\omega\, P(\omega) \exp[-i\omega\tau] . \tag{4.8}$$

The general properties of ϱ follow from the analytical behaviour of $P(\omega)$. It is immediately seen that if one could ignore the cut which starts at ω_2 $P(\omega)$ could be considered analytical in the upper half of the ω-plane. In this case we would have $\varrho(\tau) = 0$ for $\tau < 0$. There would therefore only be interaction provided that the source particle ξ_i precedes the passive particle ξ_k: $\xi_i > \xi_k$. The retardation is seen from (4.7) to be $\Delta\xi/v$. This corresponds to a situation, which can be envisaged in the following manner. The preceding particle deposits a signal on the wall. The signal is stored in the wall (without running away) and is picked up by the k-th particle as it passes. The interaction is mechanically retarded *i.e.* by $\Delta\xi/v$, where v is the velocity of the particles.

The cut which starts at ω_2 gives rise to a mechanically advanced interaction. This is of course in no contradiction with the principle of causality, since a mechanically advanced interaction can be electrically retarded ($v < c$).

The equations of motion (4.6) are valid for a linear machine. To cover the case of closed orbits, we replace $\xi_i - \xi_k$ by $\xi_i - \xi_k + nu$ in eq. (4.6) and sum over all the values of n.

To be more specific we shall limit the summation over n to $-\mathcal{N} < n < \mathcal{N}$.

The summation over n then supplies the integrand of (4.6) with a factor

$$f_{\mathcal{N}} = \sum_n \exp[i(\nu - \omega) nu/v] = \frac{\exp[i(\mathcal{N}+1)(\nu-\omega)u/v] - \exp[-i\mathcal{N}(\nu-\omega)u/v]}{\exp[i(\nu-\omega)u/v] - 1} . \tag{4.9}$$

$f_{\mathcal{N}}$ is a Dirichlet factor, which in the limit $\mathcal{N} = \infty$ has the property:

$$\lim_{\mathcal{N}\to\infty} f_{\mathcal{N}} = \omega_0 \sum_s \delta(\omega - \nu - r\omega_0) , \tag{4.10}$$

where $s = 0, \pm 1, \ldots$ and $\omega_0 = 2\pi v/u$. Using the limiting value for i given by (4.10) equation (4.6) can be replaced by

$$(-\nu^2 + \nu_0^2)\eta_k = \sum_{i,s} \omega_0 P(\nu + s\omega_0) \exp[-i(\xi_i - \xi j)s/R]\eta_i , \tag{4.11}$$

where $R = u/2\pi$. It is seen that (4.11) is obtained from (4.6) by replacing $\int \mathrm{d}\omega$ by the sum $\omega_0 \sum_s$. This takes account of the periodicity of the machine structure and it is remarkable that the frequencies $\omega_s = \nu + \omega_0 s = \omega_0(q+s)$, which appear in the argument of P are exactly those that can be revealed by a pick-up electrode placed at the circumference of the machine.

If one applies the microscopic normal mode analysis of Sect. 2 to eq. (4.11) one obtains the result

$$(-\nu^2 + \nu_0^2) Y_r = \sum N\omega_0 P(\nu + s\omega_0) g_{s-r} Y_s \,. \tag{4.12}$$

It is therefore seen that the normal modes introduced in Sect. 2 will in general not diagonalize the perturbation due to the finite resistivity of the wall. The mixing agent is represented by the Fourier coefficients g_{s-r}. Only in the absence of bunching will the normal modes of Sect. 2 coincide with the actual normal modes of the resistive-wall force.

5. – The one-beam instability for an unbunched beam.

In this case there is no mixing of the normal modes and eq. (4.12) gives because of (2.5)

$$(-\nu^2 + \nu_0^2) Y_s = N\omega_0 P(\nu + s\omega_0) Y_s \,. \tag{5.1}$$

This equation has solutions, only provided that $\nu^2 = \nu_s^2$, where

$$\nu_s^2 = \nu_0^2 - N\omega_0 P(\nu + s\omega_0) \,. \tag{5.2}$$

We first discuss the long-wave limit, *i.e.* the case of wavelengths which can be reasonably expected to be registered by a pick-up electrode capable of transmitting signals down to the cm range. An inspection of (4.3) then shows that the first term in the bracket can be neglected because of the factor $1 - v^2 = 1/\gamma^2$. For $\lambda a \ll 1$, $\partial H/\partial y$ can be approximated by $i/\pi a$. The penultimate term is negligible in comparison with the second (leading) term as long as $\nu a \ll 1$ (which is the case in all practical applications). In the last term of (4.3) we can put $M = 1$. It is then seen that also the last term can be neglected unless ω is in a small band of width $\nu(\nu a)^6$ in the neighbourhood of $\nu/2$. This can only happen if q is near a semi-integral value and we will not consider this case.

Taking only the leading term in (4.3) eq. (5.2) can now be replaced by

$$\nu_s^2 = \nu_0^2 - \frac{N r_0 v c}{2\pi\gamma a^3} Z_0 (1 + i)(q + s)^{-\frac{1}{2}} \,. \tag{5.3}$$

In this expression v is measured in cm/s, c is the velocity of light, Z_0 is defined by (3.16) with $\omega = \omega_0$; $r_0 = e^2/mc^2$ is the classical radius of the electron $= 2.8 \cdot 10^{-13}$ cm.

It is seen that it follows from eq. (5.3) that $\text{Im}\,(\nu_s^2) < 0$ provided $q+s > 0$. For $q+s < 0$ the square root has to be treated according to the prescription given in Sect. 3, *i.e.* assuming a cut extending south from $\omega = 0$. This gives $(q+s)^{-\frac{1}{2}} = -i(-(q+s))^{-\frac{1}{2}}$. We will therefore have $\text{Im}\,(\nu_0^2) > 0$ provided $q+s < 0$. Remembering that the time dependence was assumed to be $e^{-i\nu t}$ this means that we will have damping provided $q+s > 0$ and antidamping, *i.e.* a resistive-wall instability for $q+s < 0$. This is the result found by L.N.S.; it has been brillantly confirmed bu the MURA experiment.

The rise time in the case of instability is defined as $1/t_r = 2\,\text{Im}\,(\nu^2)/\nu_0 - \varrho$ (where ϱ is the damping constant of the betatron oscillations). The factor 2 is due to the fact that t_r is conventionally defined as the rise time of energy and not of amplitude. This gives because of (5.3) (neglecting ϱ)

$$t_r = \frac{\pi a^3}{N r_0 Rc} Z_0^{-1} q \sqrt{-(q+r)} \qquad \text{for } r < -q\,. \tag{5.4}$$

This is exactly the result which L.N.S. obtained for the case of circular symmetry.

The derivation of L.N.S. was based on the quasi-static approximation, which requires that the instability propagates via waves with $|\omega| \ll |k|$. In the present case this would mean $|q+s| \ll s$. In the application to the MURA experiment this condition is valid only for $s = -3$ (since $q = 2.8$). It was realized by L.N.S. by means of a control of the self-consistency of their solutions, that their result was more general than what the limitations imposed by the use of the quasi-static approximation would lead one to expect.

The present derivation is not restricted to the quasi-static approach. Indeed it can be shown that with the exception of the case $q = \frac{1}{2} + n$ $(n = 0, 1, 2, \ldots)$ the results of L.N.S. are valid for normal modes which satisfy

$$|r| \ll q/(\nu a)^2\,. \tag{5.5}$$

For the MURA experiment the right-hand side of this inequality has the value 2200.

The question naturally arises whether any significance can be attached to eq. (5.1) for frequencies which do not satisfy (5.5). It is even doubtful if all the normal modes which respect this inequality can indeed be excited. The question assumes a particular urgency if one considers that an inspection of (4.3) shows that there is another region of instability for frequencies $> \omega_2$. This instability is due to the forward wave and is associated to lifetimes of the order $t_r' \sim t_r(\nu a \gamma)^{-3}$. If this instability were real it would eliminate all hope of a cure by means of a feedback system. The considerations of Sect. 2 show, however, that the mixing time of the normal modes is of the order of ns for most of the projected machines and for ω of the order ω_2. The « infra-red » instabilities could only occur if $t'_r < T_m$. This is far from being the case for all practical applications.

In view of this we shall use in the following a simplified form of $P(\omega)$, namely

$$\omega_0 P(\omega_s) = (1+i)\, A(q+s)^{-\frac{1}{2}} \tag{5.6}$$

with

$$A = r_0 vcZ_0/2\pi a^3\gamma \quad (\text{s}^{-2})\,. \tag{5.7}$$

This will be an adequate approximation to (4.3) as long as the condition (5.5) is satisfied. The application of (5.6) will of course limit us to the treatment of phenomena which can be revealed by a r.f. pick-up electrode.

6. – Extremely bunched beams.

In this Section we consider the resistive-wall effect in the case of a single beam of N particles distributed in B equidistant bunches with N/B particles in each. The bunches have position ξ_n defined in (2.9). If the length of the bunches is very small as compared to the wavelength considered (and this is what we mean by an extremely bunched beam) we can treat all the particles of a bunch as a single oscillator of charge eN/B. Equation (4.11) can then be written in the form

$$B(-\nu^2+\nu_0^2)\eta_k = \sum_{n,s} N\omega_0 P(\nu+s\omega_0)\exp\left[-2\pi i(n-k)s/B\right]\eta_n\,. \tag{6.1}$$

For a first orientation we consider the case $B=1$. The secular equation then gives

$$\nu^2 = \nu_0^2 - N\omega_0 \sum_s P(\nu+s\omega_0)\,. \tag{6.2}$$

The sum on the right-hand side of this equation does not converge and a cut-off has to be applied at short wave lengths. However the imaginary part of the right-hand side converges and gives in the approximation (5.6)

$$\text{Im}\,(\nu^2) = -ANT(q) \tag{6.3}$$

with

$$T(q) = \sum_{s>-q} (s+q)^{-\frac{1}{2}} - \sum_{s>q} (s-q)^{-\frac{1}{2}} . \tag{6.4}$$

The following properties of the function $T(q)$ are easily verified. $T(q)$ is periodic: $T(q+1)=T(q)$. It is therefore sufficient to define $T(s)$ in the interval $0<q<1$. It is also obvious that one must have $T(1-q)=-T(q)$. (The periodicity allows one to extend the definition to negative values of q and the substitution q, $-q$ exchanges the roles of the two sums in the definition (6.4).) It follows that

$$T(\tfrac{1}{2}) = 0 \tag{6.5}$$

and it is easily seen that $T>0$ for $0<q<\frac{1}{2}$ and that it is <0 for $\frac{1}{2}<q<1$. It behaves as $(q-n)^{-\frac{1}{2}}$ in the limit $q-n \to +\varepsilon$. Its values are given in the Table:

q	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
$T(q)$	∞	4.35	2.90	2.19	1.72	1.34	1.03	0.75	0.51	0.25	0.00	-0.25

It follows from eq. (6.3) that there will be damping if $T>0$ and that there will be antidamping if $T<0$. In the case of one bunch we shall therefore have instability if

$$n+\tfrac{1}{2} < q < n+1 \qquad \text{for } n=0, 1, 2, \ldots . \tag{6.6}$$

This rule has been first discovered by COURANT. It is in agreement with the instability observed at Stanford: the rings were operated with $q \approx 0.9$.

Equations (6.3) and (6.4) allow one to calculate the rise time t_r for the unstable mode

$$t_r = \frac{\pi a^3 \gamma Z_0^{-1}}{N r_0 R c} q |T(q)|^{-1} \qquad \text{for } T<0 . \tag{6.7}$$

Comparing this result to (5.4) it will be noted that the rise times for a bunched beam are of the same order as those for an unbunched beam.

Returning to the general case of several bunches we introduce the macroscopic normal modes defined by eq. (2.11). Equation (6.1) is thus transformed into

$$(-\nu^2+\nu_0^2)H_r = N\omega_0 \sum_s P(\nu+s\omega_0) H_s G_{s-r} . \tag{6.8}$$

Using eq. (2.10) and remembering that from (2.11) it follows that $H_{s+B}=H_s$ we find that the macroscopic modes H_s are adapted to the perturbation and that one has

$$\text{(6.9)} \qquad \operatorname{Im}(\nu_r^2) = -NAT_r^B(q)\,,$$

where $T_r^B(q)$ is defined by the equation

$$\text{(6.10)} \qquad T_r^B(q) = \sum_{Bn+r>-q} (Bn+r+q)^{-\frac{1}{2}} - \sum_{Bn-r>b} (Bn-r-q)^{-\frac{1}{2}}\,.$$

Comparing this with the definition (6.4) of $T(q)$ it can at once be verified that

$$\text{(6.11)} \qquad T_r^B(q) = T((q+r)/B)B^{-\frac{1}{2}}\,.$$

To every normal mode there corresponds a reduced q: $q'=(q+r)/B$. We can therefore apply the same argument as in the case of a single bunch. A mode will be stable, if (mod (1)) the reduced q falls between 0 and $\frac{1}{2}$, and it will be unstable if q' is between $\frac{1}{2}$ and 1 (mod (1)). As an example we consider the case of Adone, with $q=3.2$ and $B=3$. We have $q'_r=0.067$, 0.40, $+0.73$ (for $r=0, 1, 2$ respectively). It follows that only the mode $r=2$ is unstable.

The stability rule for several bunches has recently and for the first time been derived by COURANT and SESSLER [8].

If the system is unstable in the s-th mode it can be easily seen that a small pick-up electrode will respond to all the frequencies $\omega_0\cdot|(q+s+B_n)|$ with $n=0, \pm1, \pm2, \ldots$. It follows that an instability of the type considered in this Section can be cured by applying a feedback on any one of these frequencies.

7. – The resistive-wall effect in the presence of two beams.

It follows from eqs. (4.2) and (4.3) that in the presence of two beams travelling in identical orbits the mechanical equations will be given by

$$\text{(7.1)} \quad (\nu_0^2-\nu^2)\eta_k^{(2)} = \ldots + \omega_0 \sum_s \hat{P}(\omega_s) \exp\left[is(\xi_k^{(2)}-\xi_i^{(1)})/R - 2i(\omega_s-\nu)t\right]\eta_i^{(1)}\,.$$

Here the indices (1) and (2) differentiate the electron beam (1) from the positron beam (2) and the dots... indicate the interaction of the positron beam with itself. The equation for the electron beam is obtained by exchanging the indices (1) and (2). $\hat{P}$ is defined by the lower sign in (4.3).

For a first orientation we shall neglect the self-interaction of the beams indicated by the dots and merely try to decide whether the interaction between two beams is at all capable of leading to a build-up of betatron oscillations. That this interaction is capable of leading to an instability by changing the optical properties of the machine has first been discussed by AMMAN and RITSON [10,11,7] whose results have been improved and refined by many others.

The mechanical problem posed by the « mutilated » equation (7.1) is different from what has been dealt with so far: the time dependence does not cancel on the right-hand side of (7.1). In addition to a possible antidamping the resistive-wall force therefore « deharmonizes » the betatron oscillations. We are not interested in the latter part of this effect though it should be pointed out that it might lead to complications, whenever the betatron oscillations are strongly anharmonic from the start.

In order to see whether there can be dedamping as a consequence of the interaction between two beams we therefore consider first the simplified model without self-interaction. Then the only term, which will give a contribution to the effect is the one with $s = 0$. This term acts only on the centre of mass of the two beams. Putting

$$N_1\eta^{(1)} = \sum_k \eta_k^{(1)} \qquad \text{and} \qquad N_2\eta^{(2)} = \sum_k \eta_k^{(2)} \tag{7.2}$$

and using the approximation (5.6) for $\hat{P}(\omega)$ (an inspection of eq. (4.3) shows that this is a good approximation also in the present case provided $\nu a \ll 1$), we get, neglecting the self-interaction,

$$\begin{cases} (\nu_0^2 - \nu^2)\eta^{(2)} = AN_1 q^{-\frac{1}{2}}(1+i)\eta^{(1)}, \\ (\nu_0^2 - \nu^2)\eta^{(1)} = AN_2 q^{-\frac{1}{2}}(1+i)\eta^{(2)}. \end{cases} \tag{7.3}$$

The secular equation has the solutions

$$\nu_\pm^2 = \nu_0^2 \pm (1+i)A\sqrt{N_1N_2/q} \tag{7.4}$$

and the adapted normal modes are

$$H_\pm = \sqrt{N_1}\,\eta^{(1)} \pm \sqrt{N_2}\,\eta^{(2)}\,. \tag{7.5}$$

---

(10) F. AMMAN and D. RITSON: *Proceedings of the International Conference on High-Energy Accelerators* (Brookhaven, 1961), p. 262.

(11) M. BASSETTI: *Calcoli numerici sugli effetti di carica spaziale in un anello di accumulazione per elettroni e positroni*, in *Laboratori Nazionali di Frascati*, LNF-62/35 (1962).

5147

Equation (7.4) cannot be taken literally, since we have only taken account of the resistive-wall force. This is quite legitimate for the imaginary part of $\hat{P}(\omega)$, but not for the real part to which a large contribution may acrue from the direct interaction between the two beams (AMMAN, RITSON) as well as from the field reflected by a perfectly conducting wall.

Taking only the imaginary part of eq. (7.4) we conclude that there is a tendency for damping of the mode H_+ and for antidamping for the mode H_-. For $N_1 \sim N_2$ $H_\pm$ may be termed the mechanical or electrical centre, respectively, of the two beams. The foregoing analysis therefore indicates that a dedamping tendency exists for the electrical centre.

Whether this tendency is actually converted into an instability depends on the self-interactions of the beams as well as on the neglected real part of either the self-interaction or the interaction between the two beams.

To see what can happen we consider the following more realistic model of a secular equation:

$$(7.6)\qquad (\nu_0^2 - \nu^2)\eta = aN_2\eta^{(2)} + \beta N_1 \eta^{(1)}$$

(+ an equation with the indices (1) and (2) interchanged). In view of (7.3) and of (6.9) we assume

$$(7.7)\qquad \begin{cases} \beta = \beta_1 + i\beta_2 = \beta_1 + iAq^{-\frac{1}{2}}\,, \\ \alpha = \alpha_1 + i\alpha_2 = \alpha_1 + iAT_0^B(q)\,. \end{cases}$$

Since the force between electron and positron is attractive and strong it appears quite possible that β_1 is very large and negative. The eigenfrequencies of the secular equation (7.6) can be easily determined and one gets

$$(7.8)\qquad 2(\nu_0^2 - \nu^2) = \alpha(N_1 + N_2) \pm (\alpha^2(N_1 - N_2)^2 + 4\beta^2 N_1 N_2)^{\frac{1}{2}}$$

and it is obvious that the imaginary part of the frequency depends not only on N_1 and N_2 but also on the real parts of α and β.

A remarkable exception is represented by the case $N_1 = N_2 = N$, for which

$$(7.9)\qquad \operatorname{Im}(\nu_\pm^2) = -AN\big(T_0^B(q) \pm q^{-\frac{1}{2}}\big)$$

and this result is completely independent of the real parts of either α or β. In this case no additional instability is created by the beam-beam interaction, provided $T_0^B(q) >$ and $q^{-\frac{1}{2}} < T_0^B(q)$. This result can be generalized to arbitrary numbers N_1, N_2 if special assumptions are made about the coefficients α and β.

5148

If it is assumed that $|\beta|_1$ is much bigger than the moduli of all the other coefficients (β_2, α_1 and α_2) we can suppress the term with α in the square root in (7.8). In this case the imaginary part of the r.h.s. of (7.8) is exclusively determined by the imaginary parts of α and β and (7.9) can be generalized to give

$$\mathrm{Im}\,(\nu^2) = -A\left(\tfrac{1}{2}(N_1+N_2)\,T_0^B(q) \pm \sqrt{N_1 N_2 q^{-\frac{1}{2}}}\right), \tag{7.10}$$

which clearly goes into (7.9) for $N_1 = N_2 = N$. Since the geometrical mean $\sqrt{N_1 N_2}$ is always less than the arithmetical mean $\frac{1}{2}(N_1+N_2)$ it follows that for an arbitrary number of particles the resistive-wall interaction of two beams does not introduce any new instabilities provided $T_0^B(q) > 0$ and $q^{-\frac{1}{2}} < T_0^B(q)$.

This is essentially the result obtained by PELLEGRINI and SESSLER (9).

8. – Summary.

In the place of a proper summary, we show the conclusions which the present work allows us to draw on the impact of the resistive-wall instability on the behaviour of one or two beams in Adone.

The essential design parameters of Adone are (12): The circumference u is 100 metres ($R = 1590$ cm). The q corresponding to the vertical betatron oscillations is assumed to be 3.2. We assume a «nominal value» of the conductivity of the wall material to be $10^{16}\ \mathrm{s}^{-1}$. Since $\omega_0 = 1.88\cdot10^7\ \mathrm{s}^{-1}$ this gives (compare (3.16)) $Z_0 = 8.65\cdot10^{-6}$. This justifies the treatment of the resistive-wall instability as a perturbation.

Equations (5.4), (6.7) and (7.9) suggest the introduction of a rate parameter ϱ_0 (s^{-1}) by means of

$$\varrho_0 = \frac{N r_0 R c Z_0}{\pi a^3 \gamma q}. \tag{8.1}$$

ϱ_0 is a characteristic rate of all the phenomena connected with the resistive-wall instability. For $2\cdot10^{11}$ particles per beam (*i.e.* a current of approximately 100 mA), $a = 4$ cm, $\gamma = 700$ (which corresponds to the phase of injection), one gets $\varrho_0 = 51.4\ \mathrm{s}^{-1}$.

---

(12) F. AMMAN, *et al.*: *Status report on the* 1.5 GeV *electron positron storage ring Adone*- in *Proceedings of the International Conference on High-Energy Accelerators* (Dubna, 1963), p. 249; *Adone, The Frascati* 1.5 GeV *electron positron storage ring,* in *Laboratori Nazionali di Frascati,* LNF-65/26 (1965); paper presented at the *Frascati International Conference on High-Energy Accelerators,* September (1965).

The results for a single beam can be summarized by stating that if the beam is split into B bunches of N/B particles each, there will be B normal modes characterized by the mode index $0 < s < B-1$. By means of

$$\varrho^s = \varrho_0 T_s^B(q) \tag{8.2}$$

with $T_s^B(q)$ defined in eq. (6.10) a rate is attributed to every mode s. Positive rates correspond to damped modes, negative rates to dedamped modes. Damping or build-up is decided by Courant's rule. We have damping if $(q+s)/B$ falls between 0 and $\frac{1}{2}$ (modulo 1) and dedamping otherwise.

The values of the function $T_s^B(q)$ can be determined by means of (6.11) from the table for $T(q) = T_0^1(q)$. For Adone one thus finds the values $T^3_{0,1,2} = 2.08,\ 0.29,\ 0.7$. The corresponding rates are

$$\varrho^{0,1,2} = 108,\ 15,\ -36\ (\mathrm{s}^{-1})\,. \tag{8.3}$$

In accordance with Courant's rule it follows that only the mode $s = 2$ is dedamped with a rise rate of 36 s^{-1}. The other two modes are damped with the damping constants defined in (8.3).

The two-beam situation can be summarized thus. The resistive-wall effect only acts in the centre of charge of either beam (*i.e.* on the modes $s = 0$). Two new modes $H_\pm$ can be derived from the centre of charge co-ordinates of the two beams. If the beams are of equal strength $N_1 = N_2 = N$ we can define the rates $\varrho^\pm$ corresponding to the two modes $H_\pm$ by means of

$$\varrho^\pm = \varrho_0 \left(T_0^B(q) \pm q^{-\frac{1}{2}}\right) \tag{8.4}$$

(compare eq. (7.9)). The upper sign corresponds to the centre of mass, the lower to the centre of charge of the two beams. In the case of Adone we have $\varrho^\pm = 136,\ 78\ \mathrm{s}^{-1}$, respectively: the centre of mass is more strongly damped than the centre of charge.

A possible mode of operation of Adone is to fill only one of the three bunches of either beam. In this case there would be no one beam instability. To cover this case $T_0^B(q)$ in (8.2) and (8.4) have to be substituted by $T(q)$. The simple beam then results stable with a damping constant of 88.5 s^{-1}. The damping constant for the mechanical and electrical modes of the two beams are respectively 117 and 59.6 s^{-1}.

It should be clear from the long list of simplifications given in the Introduction, that the numerical values for the various rates should be taken with a grain of salt.

5150

* * *

The authors wish to thank their friends who have helped them in many illuminating discussion and by the exchange of much correspondence. They ask their forgiveness if the present work does not do complete justice to their contribution. Thanks are also due to the organizers of the Novo Sibirks and Stanford Meetings, to the first for having us all impressed with the urgency of the problem, to the latter for the opportunity of solving it.

RIASSUNTO

Si discute l'instabilità trasversale causata da pareti resistive, dando speciale rilievo alle difficoltà che essa può causare nell'eseguire esperimenti con fasci incrociati. Vengono confermati i risultati ottenuti precedentemente da Laslett, Neil e Sessler [4], così come da Courant e Sessler [9] e Pellegrini e Sessler [8]. Il trattamento si basa su una geometria semplificata in cui il moto dei fasci è rettilineo. L'applicazione alla geometria circolare è resa possibile dall'introduzione di condizioni di periodicità. I risultati ottenuti sono validi sia per fasci continui sia per fasci suddivisi in pacchetti di particelle. Si discute anche la forza dovuta alla parete resistiva nel caso dei due fasci incrociati.